

# On Reduced Form Intensity-based Model with Trigger Events

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Corporate defaults may be triggered by some major market news or events such as financial crises or collapses of major banks or financial institutions. With a view to develop a more realistic model for credit risk analysis, we introduce a new type of reduced-form intensity-based model that can incorporate the impacts of both observable “trigger” events and economic environment on corporate defaults. The key idea of the model is to augment a Cox process with trigger events. Both single-default and multiple-default cases are considered in this paper. In the former case, a simple expression for the distribution of the default time is obtained. Applications of the proposed model to price defaultable bonds and multi-name Credit Default Swaps (CDSs) are provided.

**Keywords:** Reduced-Form Models; Trigger Events; Multiple Defaults; Cox Process; Defaultable Bonds; Basket Credit Default Swaps.

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# 1 Introduction

Modeling default risk has long been an important problem in both theory and practice of banking and finance. In the aftermath of the global financial crisis (GFC), much attention has been paid to investigating the appropriateness of the current practice of default risk modeling in banking, finance and insurance industries. Popular credit risk models currently used in the industries have their origins in two major classes of models. The first class of models was pioneered by Black and Scholes (1973) and Merton (1974) and is called a structural firm value model. The basic idea of the model is to describe explicitly the relationship between the asset value of a firm and the default of the firm. More specifically, the default of the firm is triggered by the event that the asset value of the firm falls below a certain threshold level related to the liabilities of the firm. The structural firm value model provides the theoretical basis for the commercial KMV model which has been widely used for default risk model in the financial industry. The second class of models was developed by Jarrow and Turnbull (1995) and Madan and Unal (1998) and is called a reduced-form, intensity-based credit risk model. The basic idea of the model is to consider defaults as exogenous events and to model their occurrences using Poisson processes and their variants. In this paper, we focus on reduced-form, intensity-based credit risk models.

Reduced-form, intensity-based credit risk models have been widely used to model portfolio credit risk and to describe dependent default risks. There are two major types of reduced-form, intensity-based models for describing dependent default risk, namely bottom-up models and top-down models. Bottom-up models focus on modeling default intensities of individual reference entities and their aggregation to form a portfolio default intensity. Some works on bottom-up models include Duffie and Garleanu (2001), Jarrow and Yu (2001), Schönbucher and Schubert (2001), Giesecke and Goldberg (2004), Duffie, Saita and Wang (2006) and Yu (2007) etc. These works differ mainly in their specifications for the parametric forms of default intensities of individual entities and the way these intensities are aggregated. The top-down models concern modeling the occurrence defaults at a portfolio level. A default intensity for the whole portfolio is modeled without reference to the identities of individual entities. Some procedures such as random thinning can be used to recover the default intensities of the individual entities. Some works on top-down models include Davis and Lo (2001), Giesecke, Goldberg and Ding (2011), Brigo, Pallavicini and Torresetti (2006), Longstaff and Rajan (2008) and Cont and Minca (2011).

We focus on the bottom-up model. Lando (1988) proposed a reduced-form, intensity-based model, where the occurrence of a default is described by the first jump of a Cox process. The main advantage of the Lando's model is that under his model, a simple pricing formula for a defaultable risky asset can be obtained. This formula is similar to the one for the default-free counterpart of the risky asset. Yu (2007) extended the

Lando’s model to incorporate multiple defaults and their correlation. The so-called “total hazard construction” by Norros (1986) and Shaked and Shathanthikumar (1987) was used to generate default times with interacting intensities. Zheng & Jiang (2009) proposed a unified factor-contagion model for modeling correlated defaults and provide an analytical solution for modeling default times with “total hazard construction”. Gu et al. (2011) introduced an “ordered default rate” method to give a recursive formula for the distribution of default times in pricing basket Credit Default Swaps (CDSs) in the context of a reduced-form, intensity-based model, which significantly enhances the computational efficiency in finding the prices of CDSs. One of the shortcomings in a number of existing reduced-form intensity-based models is that they fail to incorporate the impact of major market events, such as financial crises, on corporate defaults. This may lead to underestimation of default risk and also undervaluation of defaultable risky products.

In this paper, we address the problem of how to incorporate the direct impact of observable “trigger” events on corporate defaults in a reduced-form, intensity-based credit risk model. The key idea is to describe the occurrence of a default by the first unrecoverable “trigger” event. Armed with a Cox process for default risk, we incorporate the impact of economic environment on defaults by allowing the default intensity depending on an underlying state process representing the variation of economic environment over time. We consider both the single-default and multiple-default cases. In the single-default case, we obtain a simple expression for the distribution of the default time. This distribution is useful for pricing defaultable securities. We then extend the model to the multiple-default case with a view to incorporating default correlation. To provide a tractable and practical way to value defaultable securities, we focus on the case where the state process for economic environment is modeled by a continuous-time, finite-state, observable Markov chain. Applications of the proposed model to value defaultable bonds and basket Credit Default Swaps (CDSs) are discussed. We also provide numerical results to illustrate the sensitivity of the prices of these securities with respect to changes in key parameters.

The rest of the paper is organized as follows. Section 2 presents the basic model and derives the distribution of a default time. Section 3 provides the extension of model framework to the case of multiple correlated defaults. Section 4 presents the Markov chain model for the state process of economic environment. Applications to pricing the defaultable securities are given in Section 5. We then conclude the paper in Section 6.

## 2 The Basic Model

A popular reduced-form, intensity-based credit risk model was proposed by Lando (1998), where the occurrence of a default was described by the first jump of a Cox process with stochastic intensities  $\{\lambda_t\}_{t \geq 0}$  depending on an underlying state process  $\{X_t\}_{t \geq 0}$  describing the evolution of economic environment over time. Here we aim at extending the Lando’s

model by incorporating explicitly the “trigger” events such as financial crises or extraordinary market news into default risk modeling. We assume that these “trigger” events are observable and may lead to the default of a corporation. Furthermore, we suppose that the corporation may recover from a “trigger” event via re-organizing its resources or re-structuring. We assume that the emergence of “trigger” events are modeled by a Cox process, which is also called doubly Poisson process in the statistical literature and has a remarkable history in statistics. In what follows, we shall describe the mathematical set up of the Cox process describing the “trigger” events.

Uncertainty is described by a complete filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ , where  $P$  is a given probability measure<sup>1</sup> and  $\{\mathcal{F}_t\}_{t \geq 0}$  is a filtration satisfying some usual conditions, (i.e., the right-continuity and the  $P$ -completeness). We shall define precisely the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  in later part of this section.

To describe the evolution of the state of economic environment over time, we define a state process  $\{X_t\}_{t \geq 0}$ . We assume that  $\{X_t\}_{t \geq 0}$  is a càdlàg, (i.e., right continuous with left limits), process on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$  with state space  $\mathfrak{R}$ . Let  $\{N_t\}_{t \geq 0}$  be a standard Poisson process on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ , with  $N_0 = 0$ ,  $P$ -a.s. Write  $\{\lambda_t\}_{t \geq 0}$  for a bounded, non-negative stochastic process on  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ . We assume that for each  $t \geq 0$ ,  $\lambda_t := \lambda(X_t)$ , for some non-negative continuous function  $\lambda$  and that  $\{N_t\}_{t \geq 0}$  and  $\{\lambda_t\}_{t \geq 0}$  are stochastically independent under  $P$ . For each  $t \geq 0$ , we define the following cumulative process  $\{\Lambda_t\}_{t \geq 0}$ :

$$\Lambda_t := \int_0^t \lambda_s ds < \infty .$$

Then a Cox (point) process  $\{\tilde{N}_t\}_{t \geq 0}$  with intensity measure  $\Lambda := \{\Lambda_t\}_{t \geq 0}$ <sup>2</sup> is defined by:

$$\tilde{N}_t := N_{\Lambda_t} .$$

For each  $i = 1, 2, \dots$ , let  $\tau^i$  be the arrival time of the  $i^{th}$  “trigger” event, which is modeled as the arrival time of a jump in the Cox process. Once a “trigger” event occurs at time  $s$ , a loss occurs to the firm, which is modeled as an arbitrary independent random variable “ $L$ ”. We write  $\{C_t\}_{t \geq 0}$  for the process depending on the state process  $\{X_t\}_{t \geq 0}$ , where

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<sup>1</sup> When we wish to evaluate the risk of a credit portfolio, we need to use a real-world probability measure. In this case,  $P$  can be interpreted as a real-world probability measure. On the other hand, when we wish to price defaultable securities, we must use a risk-neutral probability measure. In this case, there are two approaches to interpret  $P$ . The first approach is to interpret  $P$  as a risk-neutral probability measure and start with the risk-neutral probability measure directly. The second approach is to interpret  $P$  as a real-world probability measure and then use a measure change for Poisson processes to transform the real-world probability measure to a risk-neutral one. To simplify our discussion, when we discuss the pricing of defaultable securities, we shall adopt the first approach.

<sup>2</sup> Strictly speaking, the intensity measure  $\Lambda$  is defined on the  $\sigma$ -field generated by bounded subsets of the interval  $[0, \infty)$ . The intensity measure  $\Lambda$  also has a Random density, or Radon-Nikodym derivative,  $\lambda$ , by definition.

$C_t := C(X_t)$  as the threshold value at time  $t$ , for some non-negative continuous function  $C$ . If  $L \leq C_s$ , then the firm can recover from the “trigger” event; otherwise, the firm defaults. Davis & Lo (2001) introduce a Bernoulli contagion variable in a homogeneous setup, which is similar to variable “ $L$ ” Here.

Let  $\{\tau^i\}_{i=1,2,\dots}$  be a sequence of stopping times representing the arrival times of “trigger” events defined by

$$\tau^i := \inf\{t \geq 0 : \tilde{N}_t \geq i\} ,$$

and  $\{L^i\}_{i=1,2,\dots}$  a sequence of arbitrary independent and identically distributed random variables. We assume that  $\{\tau^i\}_{i=1,2,\dots}$  and  $\{L^i\}_{i=1,2,\dots}$  are stochastically independent under  $P$ . Define a random variable  $K$  taking values in  $\{1, 2, \dots\}$  by

$$K := \min\{i : L^i > C_{\tau^i}\} .$$

Then the default time of a firm  $\tau$  is defined by:  $\tau := \tau^K$ . We now specify more explicitly the information structure of our model. Define the filtrations  $\{\mathcal{G}_t\}_{t \geq 0}$ ,  $\{\mathcal{H}_t\}_{t \geq 0}$  and  $\{\mathcal{I}_t\}_{t \geq 0}$  as follow:

$$\mathcal{G}_t := \sigma\{X_s : 0 \leq s \leq t\} \vee \mathcal{N} ,$$

$$\mathcal{H}_t := \sigma\{\tilde{N}_s : 0 \leq s \leq t\} \vee \mathcal{N} ,$$

and

$$\mathcal{I}_t := \sigma\{1_{\{\tau \leq s\}} : 0 \leq s \leq t\} \vee \mathcal{N} ,$$

where  $\mathcal{N}$  is the collection of all  $P$ -null subsets in  $\mathcal{F}$  and  $1_A$  is the indicator function of an event  $A$ . Here we assume that the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  is specified as follows:  $\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t \vee \mathcal{I}_t$ . This represents the full observable information structure in our model.

For each  $s \geq 0$ , let  $q_s$  be the probability that the firm can recover from a “trigger” event if the event occurs at time  $s$  given the underlying state  $X_s$ . Then,

$$q_s := P(L \leq C(X_s) \mid X_s).$$

Let  $p_s := 1 - q_s$ . Then

$$p_s = P(L > C(X_s) \mid X_s).$$

The following result is one of our main results which gives the conditional and unconditional distributions of the default time  $\tau$ . The proof can be found the Appendix.

**Proposition 1** *For any  $t \geq s > 0$ ,*

$$P(\tau > s \mid \mathcal{G}_t) = \exp \left\{ - \int_0^s p_u \lambda_u du \right\}$$

and

$$P(\tau > s) = E \left[ \exp \left\{ - \int_0^s p_u \lambda_u du \right\} \right] ,$$

where  $E$  is an expectation under  $P$ .

The following results are also important to characterize the probability laws of the default time  $\tau$ . Their proofs can be found in the Appendix.

**Lemma 1** *For any  $s < t$ ,*

$$E(1_{\{\tau > t\}} \mid \mathcal{G}_t \vee \mathcal{H}_s \vee \mathcal{I}_s) = 1_{\{\tau > s\}} \exp \left\{ - \int_s^t p_u \lambda_u du \right\}.$$

**Lemma 2** *The process*

$$1_{\{t \geq \tau\}} - \int_0^t p_u \lambda_u 1_{\{u < \tau\}} du, \quad t \geq 0,$$

*is an  $(\{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})$ -martingale.*

To price the defaultable securities in the proposed model as above, we construct three “building blocks” as in Lando(1988). Before that, we assume that all of the expectations in this paper are taken under an equivalent martingale measure. We suppose that  $T$  denotes the expiry date of all contingent claims. The three “building blocks” are as follows:

- (I)  $X 1_{\{\tau > T\}}$ : A payment  $X \in \mathcal{G}_T$  at a fixed date  $T$  which occurs if there has been no default before time  $T$ .
- (II)  $Y_s 1_{\{\tau > s\}}$ : A stream of payments at a rate specified by the  $\{\mathcal{G}_t\}_{t \geq 0}$ -adapted process  $Y$  which stops when default occurs.
- (III)  $Z_\tau$ : A recovery payment at the time of default, where  $Z$  is a  $\{\mathcal{G}_t\}_{t \geq 0}$ -adapted process.

Now we proceed to give the pricing formula of these three “building blocks”.

**Proposition 2** *Suppose*

$$\begin{aligned} & \exp\{-\int_0^T r_s ds\} X, \\ & \int_0^T Y_s \exp\{-\int_0^s r_u du\} ds, \\ & \int_0^T Z_s \lambda_s p_s \exp\{-\int_0^s (r_u + \lambda_u p_u) du\} ds \end{aligned}$$

*are integrable random variables. Then,*

$$E(\exp\{-\int_0^T r_s ds\} X 1_{\{\tau > T\}}) = E(\exp\{-\int_0^T (r_s + p_s \lambda_s) ds\} X), \quad (1)$$

$$E(\int_0^T Y_s 1_{\{\tau > s\}} \exp\{-\int_0^s r_u du\} ds) = E(\int_0^T Y_s \exp\{-\int_0^s (r_u + p_u \lambda_u) du\} ds) \quad (2)$$

*and*

$$E(\exp\{-\int_0^\tau r_s ds\} Z_\tau) = E(\int_0^T Z_s \lambda_s p_s \exp\{-\int_0^s (r_u + \lambda_u p_u) du\} ds). \quad (3)$$

**Proof:** The results follow directly from Proposition 1. □

### 3 An Extension to Dependent Multiple defaults

In this section, we shall extend the basic model in the last section to the multiple-default case. Again we consider the filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ , where the filtration satisfies the usual conditions and is specified as in the last section. Here the underlying state process of economic environment  $\{X_t\}_{t \geq 0}$  is also defined as in the last section, (i.e., a càdlàg process). For each  $i = 1, 2, \dots, n$ , let  $\tau_i$  be a stopping time representing the default time of the  $i^{th}$  individual obligor.

Let  $\{N_t^i\}_{t \geq 0}$ ,  $i = 1, 2, \dots, n$ , be  $n$  independent standard Poisson processes with  $N_0^i = 0$ ,  $P$ -a.s. Write, for each  $i = 1, 2, \dots, n$ ,  $\{\lambda_t^i\}_{t \geq 0}$  for a bounded, non-negative stochastic process which is adapted to the enlarged filtration containing the filtration  $\{\mathcal{G}_t\}_{t \geq 0}$  and the filtration generated by the sequence of stopping times  $\{\tau_i\}_{i=1,2,\dots,n}$ , where the latter filtration will be defined precisely later in this section.

For each  $i = 1, 2, \dots, n$  and each  $t \geq 0$ , we define the intensity measure  $\Lambda^i := \{\Lambda_t^i\}_{t \geq 0}$  for the  $i^{th}$ -individual obligor by putting:

$$\Lambda_t^i := \int_0^t \lambda_s^i ds < \infty .$$

Then for each  $i = 1, 2, \dots, n$ , a Cox (point) process  $\{\tilde{N}_t^i\}_{t \geq 0}$  for the  $i^{th}$  obligor associated with the intensity measure  $\Lambda_i$  is defined by:

$$\tilde{N}_t^i := N_{\Lambda_t^i}^i .$$

Define the arrival time of  $j^{th}$  “trigger” event of name  $i$ ,

$$\tau_i^j = \inf\{t : \tilde{N}_t^i \geq j\}, i = 1, 2, \dots, n.$$

Consider an array of i.i.d. arbitrary random variables

$$\{L_i^j, i = 1, 2, \dots, n, j = 1, 2, \dots\}.$$

We suppose that this array of random variables is independent of all previous events. Define the random variables:

$$K_i := \min\{j : L_i^j > C_{\tau_i^j}^j\} , \quad i = 1, 2, \dots, n,$$

where  $C_s^i := C^i(X_s)$  is the threshold value of name  $i$  at time  $s$  and  $C^i$  a non-negative continuous function for  $i = 1, 2, \dots, n$ .

Then the default time of name  $i$  is defined as

$$\tau_i := \tau_i^{K_i}.$$

The information structure of the multiple-default model is specified as follows:

$$\mathcal{H}_t := \sigma\{(\tilde{N}_s^i)_{i=1}^n : 0 \leq s \leq t\} \vee \mathcal{N} ,$$

and

$$\mathcal{I}_t := \sigma\{(1_{\{\tau_i \leq s\}})_{i=1}^n : 0 \leq s \leq t\} \vee \mathcal{N}.$$

Again we assume that

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t \vee \mathcal{I}_t.$$

We also define

$$p_s^i := P(L > C^i(X_s) \mid X_s), \quad q_s^i := P(L \leq C^i(X_s) \mid X_s).$$

We further assume that for each  $i = 1, 2, \dots, n$ ,  $P(0 < \tau_i < \infty) = 1$  and  $P(\tau_i = \tau_j) = 0$  for any  $i \neq j$ . Moreover,  $X$  is an “exogenous” stochastic process in the sense that  $\mathcal{G}_\infty$  and  $\mathcal{H}_t \vee \mathcal{I}_t$  are conditionally independent given  $\mathcal{G}_t$ .

We remark that this framework for multi-name defaults allows the default intensities to be sensitive to the observed defaults as well as the underlying state process  $X$ . Taking the consideration of the real practice that once a firm recovers from a “trigger” event, the impact of such a “trigger” is restricted to a very minor level,  $\{\lambda_t\}_{t \geq 0}$  is supposed  $\{\mathcal{G}_t \vee \mathcal{I}_t\}_{t \geq 0}$ -adapted.

For each  $i = 1, 2, \dots, n$ , if we treat information about the state process  $X$  and observed defaults of other names up to time  $t$  as a new “ $\mathcal{G}_t$ ”, then applying Lemma 2 in the single-name default case, we deduce that the process defined by:

$$1_{\{t \geq \tau_i\}} - \int_0^t p_u^i \lambda_u^i 1_{\{u < \tau_i\}} du, \quad t \geq 0,$$

is an  $(\{\mathcal{F}_t\}_{t \geq 0}, P)$ -martingale.

Consequently, the total hazard construction method, Yu(2007) and Zhang & Jiang (2009), the order default rate approach by Gu et al.(2011) can be applied to compute the multi-name default time distribution under this framework. We also remark that this framework is indeed a generalization of the standard reduced-form intensity-based model by Lando (1988) and Yu (2007). If we set  $p_s^i \equiv 1$ , i.e., the firm cannot recover from “trigger” event  $P$ -a.s., then our proposed model is reduced to be the standard reduced-form intensity-based model.

## 4 State Process as an Observable Markov Chain

Suppose now that the state process  $(X_t)_{t \geq 0}$  follows a continuous-time, homogeneous,  $M$ -state Markov chain on  $(\Omega, \mathcal{F}, P)$  with state space  $\{x_1, x_2, \dots, x_M\}$ . The amount of time the chain  $X$  spends in state  $x_i$  before making a transition into another state is exponentially distributed with rate  $v_i$ . When the process leaves state  $x_i$ , it next enters state  $x_j$  with probability  $p_{ij}$ , where

$$p_{ii} = 0 \quad \text{and} \quad \sum_j p_{ij} = 1$$



for all  $i$ .

Let  $T_{ij}(t)$  be the occupation time of the chain  $X$  in state  $x_j$  in the time interval  $[0, t]$  starting from  $X_0 = x_i$ . We wish to determine the joint distribution of  $(T_{i1}(t), T_{i2}(t), \dots, T_{iM}(t))$ . Note that the joint distribution of  $(T_{i1}(t), T_{i2}(t), \dots, T_{iM}(t))$  is completely determined by its joint moment generating function. We shall derive the joint moment generating function in the sequel.

For each  $i = 1, 2, \dots, M$ , let

$$T_i(t) = (T_{i1}(t), T_{i2}(t), \dots, T_{iM}(t))^T$$

and

$$u = (u_1, u_2, \dots, u_M)^T \in \mathcal{R}^M.$$

The moment generating function of  $T_i(t)$  is given by:

$$\Psi_i(u, t) = E(\exp\{u^T T_i(t)\})$$

Let  $\xi_i$  denote the time of the first jump from state  $x_i$  to another state, i.e.,  $\xi_i \sim \exp(v_i)$ . Hence

$$\begin{aligned} \Psi_i(u, t) &= E(\exp\{u^T T_i(t)\}) \\ &= E(E(\exp\{u^T T_i(t)\} \mid \xi_i)) \\ &= \sum_{k \neq i} p_{ik} v_i \int_0^t e^{(u_i - v_i)s} E(\exp\{u^T T_k(t - s)\}) ds + e^{(u_i - v_i)t} \\ &= \sum_{k \neq i} p_{ik} v_i \int_0^t e^{(u_i - v_i)s} \Psi_k(u, t - s) ds + e^{(u_i - v_i)t} \\ &= \sum_{k \neq i} p_{ik} v_i \int_0^t e^{(u_i - v_i)(t-s)} \Psi_k(u, s) ds + e^{(u_i - v_i)t}. \end{aligned}$$

Taking the partial derivative with respect to  $t$  on both sides yields,

$$\frac{\partial}{\partial t} \Psi_i(u, t) = \sum_{k \neq i} p_{ik} v_i \Psi_k(u, t) + (u_i - v_i) \Psi_i(u, t). \quad (4)$$

For simplicity, we write

$$\Psi_u(t) = (\Psi_1(u, t), \Psi_2(u, t), \dots, \Psi_M(u, t))^T$$

and define the matrix

$$A = \begin{bmatrix} u_1 - v_1 & p_{12}v_1 & \cdots & p_{1M}v_1 \\ p_{21}v_2 & u_2 - v_2 & \cdots & p_{2M}v_2 \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1}v_M & p_{M2}v_M & \cdots & u_M - v_M \end{bmatrix}$$

Hence Equation (4) can be rewritten as a system of homogeneous linear differential equations with constant coefficients

$$\Psi_u'(t) = A \Psi_u(t), \quad (5)$$

where  $\Psi_u(0) = \mathbf{1}$ . Solving the ODEs, by using the Fundamental Theorem for Linear Systems (see Chapter 1 in L. Perko (2001)), yields the following proposition.

**Proposition 3** *The moment generating function of  $T_i(t)$  is given by  $\Psi_i(u, t)$ , where*

$$\Psi_u(t) = (\Psi_1(u, t), \Psi_2(u, t), \dots, \Psi_M(u, t))^T$$

*has a unique solution as*

$$\Psi_u(t) = e^{At} \mathbf{1}.$$

This result will be used to price basket credit default swaps under our proposed model in Section 5.2.

## 5 Applications

In this section, we apply the ordered default rate approach in Gu et al. (2011) to price different defaultable securities under various assumptions of default correlation structures.

### 5.1 Defaultable Bonds

We first discuss the pricing of defaultable zero-coupon bonds with zero recovery under a constant default-free rate  $r$  in the case of two firms. The defaultable bond price is then proportional to the conditional survival probability  $P(\tau_i > T \mid \mathcal{F}_t)$ . A “looping default” case with two firms was presented by Jarrow and Yu (2001), where they assumed that the default times of the two firms  $\tau^A$  and  $\tau^B$  have the following intensities:

$$\lambda_t^A = a_1 + a_2 1_{\{t \geq \tau^B\}}$$

and

$$\lambda_t^B = b_1 + b_2 1_{\{t \geq \tau^A\}}.$$

Yu (2007) gave the default time distribution of this two-firm case in the standard reduced-form intensity-based modelling framework. In this section, we provide a solution of the two-firm case in our proposed modeling framework, where  $p_t^i \equiv p$ . In other words, the firm recover from a “trigger” event with a constant probability  $1 - p$ .

Let  $\tau^A \wedge \tau^B$  denote the first default time of these two names, the the default rate of  $\tau^A \wedge \tau^B$  is  $a_1 + b_1$ . By Proposition 1,

$$P(\tau^A \wedge \tau^B > t) = \exp(-p(a_1 + b_1)t).$$

Hence

$$\begin{cases} P(\tau^A > t, \tau^A < \tau^B) = \frac{a_1}{a_1 + b_1} e^{-p(a_1 + b_1)t} \\ P(\tau^B > t, \tau^A > \tau^B) = \frac{b_1}{a_1 + b_1} e^{-p(a_1 + b_1)t} \end{cases}$$

and also

$$\begin{cases} P(\tau^A - \tau^B > t \mid \tau^A > \tau^B) = e^{-p(a_1 + a_2)t} \\ P(\tau^B - \tau^A > t \mid \tau^B > \tau^A) = e^{-p(b_1 + b_2)t}. \end{cases}$$

Then by making use of convolution, we have

$$\begin{cases} \frac{\partial}{\partial t} P(\tau^A \leq t, \tau^A > \tau^B) = \frac{b_1(a_1 + a_2)p}{b_1 - a_2} (e^{-p(a_1+a_2)t} - e^{-p(a_1+b_1)t}) \\ \frac{\partial}{\partial t} P(\tau^B \leq t, \tau^B > \tau^A) = \frac{a_1(b_1 + b_2)p}{a_1 - b_2} (e^{-p(b_1+b_2)t} - e^{-p(a_1+b_1)t}) \end{cases}$$

Therefore, the marginal density of  $\tau^A$  and  $\tau^B$  are given by

$$\begin{cases} f_{\tau^A}(t, p) = \frac{b_1(a_1 + a_2)p}{b_1 - a_2} (e^{-p(a_1+a_2)t} - e^{-p(a_1+b_1)t}) + a_1 p e^{-p(a_1+b_1)t} \\ f_{\tau^B}(t, p) = \frac{a_1(b_1 + b_2)p}{a_1 - b_2} (e^{-p(b_1+b_2)t} - e^{-p(a_1+b_1)t}) + b_1 p e^{-p(a_1+b_1)t}. \end{cases}$$

Comparing the results in Yu (2007), the marginal density of  $\tau^A$  and  $\tau^B$  under the standard reduced-form intensity-based model are, respectively, given by:

$$g_{\tau^A}(t) = f_{\tau^A}(t, 1) \quad \text{and} \quad g_{\tau^B}(t) = f_{\tau^B}(t, 1).$$

Hence as the remark in Section 3, our proposed model is a generalization of the standard one.

## 5.2 Basket Credit Default Swaps

In this section we discuss the pricing of a basket CDS in the context of the multiple-default model described in Section 3 and provide the sensitivity analysis of the CDS premium by varying key model parameters. We consider a basket CDS contract that pays \$1 if  $k$ th-to-default out of a portfolio of reference entities occurs prior to expiry date. To simplify our discussion, we assume that the payment (if any) occurs at expiration, and that the buyer pays a premium at the initiation of the swap contract. With a constant risk-free interest rate  $r$ , the premium on the  $k$ th-to-default CDS is given by:

$$S_k = \exp\{-rT\} P(\tau^k \leq T),$$

where  $\tau^k$  is the  $k$ th-to-default time and  $T$  is the expiry date.

We assume the following default intensity for  $n$  reference names:

$$\lambda_t^i = X_t(1 + b \sum_{j \neq i} 1_{\{\tau_j \leq t\}}), \quad i = 1, 2, \dots, n,$$

and for all  $i$ ,  $p_t^i = 1 - e^{-cX_t}$ .

Let  $\tau^k$  denote the  $k$ th default time and  $\tau^0$  is assigned to be 0. Let

$$\lambda_u^{(k)} = X_u(1 + b(k-1))(n - (k-1))$$

denote the  $k$ th-to-default rate as in Gu et al. (2011). By Proposition 1, we have the following relations:

$$P(\tau^k - \tau^{k-1} > s \mid \sigma(\tau^{k-1}) \vee \mathcal{G}_\infty) = \exp \left\{ - \int_{\tau^{k-1}}^{\tau^{k-1} + s} \lambda_u^{(k)} p_u du \right\}.$$

Hence we have

$$f_{\tau^k - \tau^{k-1} | \tau^{k-1}, \mathcal{G}_\infty}(s) = \lambda_{\tau^{k-1} + s}^{(k)} p_{\tau^{k-1} + s} \exp \left\{ - \int_{\tau^{k-1}}^{\tau^{k-1} + s} \lambda_u^{(k)} p_u du \right\}.$$

Then by convolution, given  $\mathcal{G}_\infty$ , we have the following recursive formula for the conditional density function of  $\tau^k$ :

$$f_{\tau^k | \mathcal{G}_\infty}(t) = \lambda_t^{(k)} p_t \int_0^t f_{\tau^{k-1} | \mathcal{G}_\infty}(u) \exp \left\{ - \int_u^t \lambda_s^{(k)} p_s ds \right\} du.$$

Let  $Y_t = X_t p_t$  and by the recursive formula, assuming  $b \neq 1/i$  for all  $i = 1, 2, \dots, n-1$ , the PDF and CDF of  $\tau^k$  respectively, given the evolution of  $(X_t)_{t \geq 0}$ , are given by

$$f_{\tau^k | \mathcal{G}_\infty}(t) = \sum_{j=0}^{k-1} \alpha_{k,j} Y_t \exp \left\{ -\beta_j \int_0^t Y_u du \right\}$$

$$P(\tau^k \leq t | \mathcal{G}_\infty) = \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{\beta_j} (1 - \exp \{ -\beta_j \int_0^t Y_u du \})$$

where the coefficients are given by the following recursion:

$$\begin{cases} \alpha_{k+1,j} = \begin{cases} \frac{\alpha_{k,j} \beta_k}{\beta_k - \beta_j}, & j = 0, 1, \dots, k-1 \\ - \sum_{u=0}^{k-1} \alpha_{k+1,u}, & j = k \end{cases} \\ \beta_j = (n-j)(1+jb) \end{cases}$$

where  $\alpha_{1,0} = n$ . Hence,

$$f_{\tau^k}(t) = \sum_{j=0}^{k-1} \alpha_{k,j} E(Y_t \exp \{ -\beta_j \int_0^t Y_u du \})$$

$$P(\tau^k \leq t) = \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{\beta_j} (1 - E(\exp \{ -\beta_j \int_0^t Y_u du \})).$$

If we assume  $(X_t)_{t \geq 0}$  is a continuous time time-homogeneous Markov chain having  $M$  states  $\{x_1, x_2, \dots, x_M\}$ . By using the results in Section 4, we find the swap premium  $S_k$ . Indeed,  $Y_t = X_t p_t$  is also a continuous time time-homogeneous Markov chain having  $M$  states  $\{y_1, y_2, \dots, y_M\}$ , where  $y_i = x_i(1 - e^{-cX_i})$ . Then starting from state  $x_i$ , let  $y = (y_1, y_2, \dots, y_M)^T$ , we have

$$\begin{aligned} P(\tau^k \leq t) &= \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{\beta_j} (1 - E(\exp \{ -\beta_j y^T T_i(t) \})) \\ &= \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{\beta_j} (1 - \Psi_i(-\beta_j y, t)). \end{aligned}$$

Consequently we have

$$S_k = e^{-rT} \sum_{j=0}^{k-1} \frac{\alpha_{k,j}}{\beta_j} (1 - \Psi_i(-\beta_j y, T)).$$

To give an example of the pricing of basket CDS, we set the number of states of  $(X_t)_{t \geq 0}$ ,  $M = 4$ , the states  $x_i = 0.1i$ ,  $p_{ij} = 1/3$  for any  $i \neq j$ ,  $v = (v_1, v_2, v_3, v_4) = (3, 2, 1, 3)$ . We remark that the four states can represent the macroeconomic environment as “prosperous”, “good”, “neutral”, “bad”. We assume there are 10 firms in the industry ( $n = 10$ ) and we start from state  $x_1$ , ie,  $X_0 = x_1$ , the default free interest rate  $r = 0.05$ , the expiry date  $T = 5$  years. To examine the effect of contagion parameters  $b$  and  $c$ , we present the swap premium with the contagion parameters varying.

As shown in Figure 1, an increase in the parameter  $b$  indeed raises the premium for all  $k$  as our intuition while the first-to-default swap price remains unchanged due to the contagion has no effect for the first-to-default time. The premium increases as the parameter  $c$  becomes bigger since the recovery from a “trigger” becomes more difficult in a weak macroeconomic environment.

## 6 Concluding Remark

In this paper, we introduce an extended reduced-form intensity-based model with “trigger” events, which captures an important feature in the market, namely the observable events that trigger defaults. Furthermore, we extend the model into a multi-name default model, with intensities driven by the history of the exogenous state process representing the macroeconomic environment and the observed defaults. A Markov chain model for the state process is also proposed to model the macroeconomic environment and the distribution of the occupation time of each states is deduced by solving a system of homogeneous ODEs. We demonstrate the pricing of two defaultable contingent claims using the ordered default rate approach by Gu et al. (2011) with numerical examples.

There are still many outstanding issues for further research such as multi-state migration model that links loss sizes of a firm due to “trigger” events and its credit states which in turn affect payoffs to investors, and hybrid model of “trigger” events and default contagion with effective computation. We are currently investigating these problems.

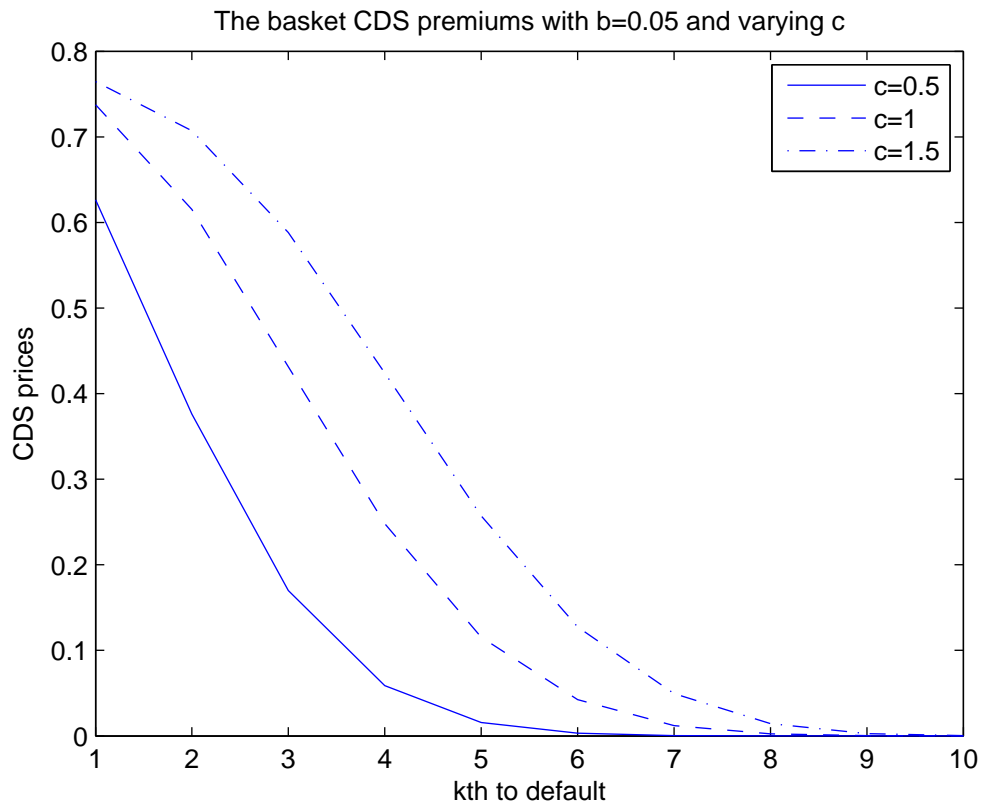
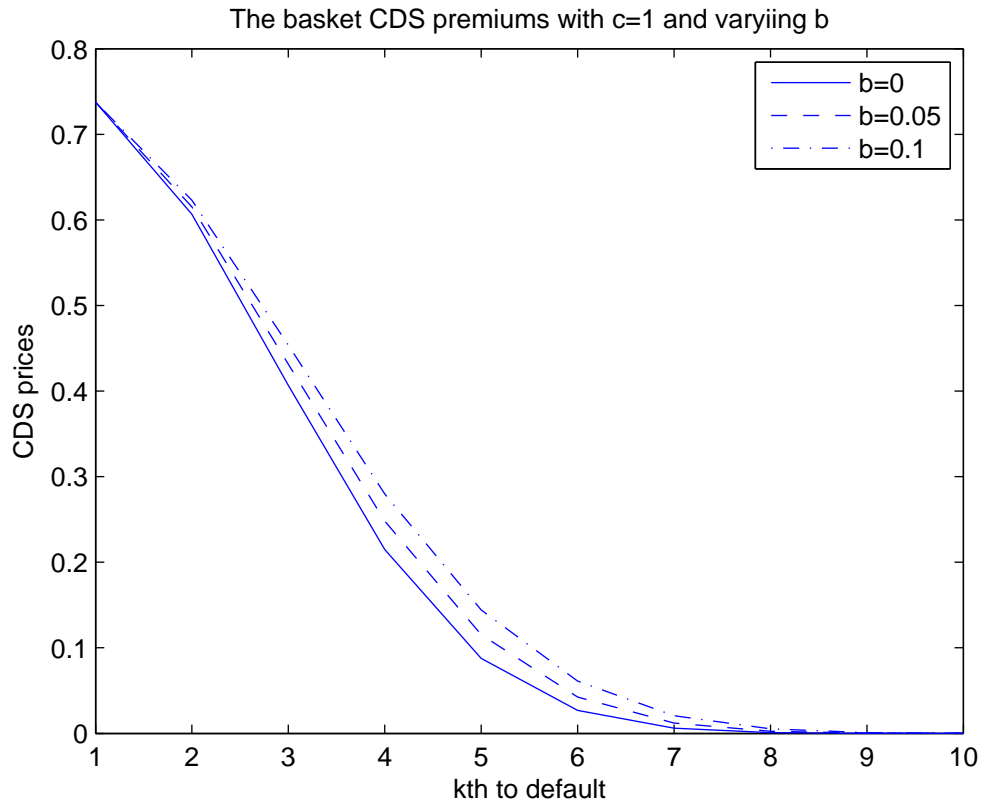


Figure 1: The effect of the contagion parameters on the basket CDS premiums

## 7 Appendix

### 7.1 Proof of Proposition 1

**Proof:** Note that

$$\begin{aligned}
P(\tau \leq s \mid \mathcal{G}_t) &= \sum_{i=1}^{\infty} P(\tau \leq s, K = i \mid \mathcal{G}_t) \\
&= \sum_{i=1}^{\infty} E[E[1_{\{\tau_i \leq s\}} 1_{\{K=i\}} \mid \sigma(\tau^1, \tau^2, \dots, \tau^i) \vee \mathcal{G}_{\infty}] \mid \mathcal{G}_t] \\
&= \sum_{i=1}^{\infty} E[1_{\{\tau^i \leq s\}} p_{\tau^i} q_{\tau^i-1} q_{\tau^i-2} \dots q_{\tau^1} \mid \mathcal{G}_t].
\end{aligned}$$

By definition,

$$P(\tau^{j+1} - \tau^j > u \mid \sigma(\tau^j) \vee \mathcal{G}_{\infty}) = \exp\left\{-\int_{\tau^j}^{\tau^j+u} \lambda_v dv\right\}$$

which implies that

$$\frac{\partial}{\partial u} P(\tau^{j+1} - \tau^j > u \mid \sigma(\tau^j) \vee \mathcal{G}_{\infty}) = -\lambda_{\tau^j+u} \exp\left\{-\int_{\tau^j}^{\tau^j+u} \lambda_v dv\right\}.$$

Hence the joint density function of  $\{\tau^1, \tau^2, \dots, \tau^i\}$  given  $\mathcal{G}_{\infty}$  is:

$$f_{\tau^1, \tau^2, \dots, \tau^i \mid \mathcal{G}_{\infty}}(t_1, t_2, \dots, t_i) = \lambda_{t_1} \lambda_{t_2} \dots \lambda_{t_i} \exp\left\{-\int_0^{t_i} \lambda_v dv\right\}$$

Let  $\{Z^j\}_{j=0}^{\infty}$  be a sequence of continuous stochastic processes  $Z^j := \{Z_u^j\}_{u \geq 0}$  such that

$$Z_u^{j+1} := \int_0^u Z_v^j \lambda_v q_v dv,$$

where  $Z_u^0 = 1$ . Thus

$$\begin{aligned}
&E[1_{\{\tau^i \leq s\}} p_{\tau^i} q_{\tau^i-1} q_{\tau^i-2} \dots q_{\tau^1} \mid \mathcal{G}_{\infty}] \\
&= \int_0^s \int_0^{t_i} \int_0^{t_{i-1}} \dots \int_0^{t_2} q_{t_1} q_{t_2} \dots q_{t_{i-1}} p_{t_i} f_{\tau^1, \tau^2, \dots, \tau^i \mid \mathcal{G}_{\infty}}(t_1, t_2, \dots, t_i) dt_1 dt_2 \dots dt_i \\
&= \int_0^s \lambda_u p_u \exp\left\{-\int_0^u \lambda_v dv\right\} Z_u^{i-1} du.
\end{aligned}$$

Let  $H_u = \sum_{i=1}^{\infty} Z_u^i$ . Then by the monotone convergence theorem,

$$H_u = \int_0^u \left(\sum_{i=1}^{\infty} Z_v^i + 1\right) \lambda_v q_v dv = \int_0^u (H_v + 1) \lambda_v q_v dv.$$

Note that a càdlàg process has at most countably many discontinuities on a compact interval. Since  $\{\lambda_u\}_{u \geq 0}$  and  $\{p_u\}_{u \geq 0}$  are càdlàg processes,  $H_u$  is continuous in  $u \in \mathfrak{R}_+$  and differentiable except on a set with countable points which has measure zero. Consequently, except on an evanescent set,  $H_u$  satisfies the following Ordinary Differential Equation (ODE):

$$\frac{dH_u}{du} = (H_u + 1) \lambda_u q_u,$$

and  $H_0 = 0$ .

Solving the ODE yields:

$$H_u = \exp \left\{ \int_0^u \lambda_v q_v dv \right\} - 1.$$

Therefore,

$$\begin{aligned} P(\tau \leq s \mid \mathcal{G}_t) &= \sum_{i=1}^{\infty} E[E[1_{\{\tau^i \leq s\}} p_{\tau^i} q_{\tau^{i-1}} q_{\tau^{i-2}} \dots q_{\tau^1} \mid \mathcal{G}_{\infty}] \mid \mathcal{G}_t] \\ &= \sum_{i=1}^{\infty} E[\int_0^s Z_u^{i-1} \lambda_u p_u \exp\{-\int_0^u \lambda_v dv\} du \mid \mathcal{G}_t] \\ &= E[\int_0^s (1 + H_u) \lambda_u p_u \exp\{-\int_0^u \lambda_v dv\} du \mid \mathcal{G}_t] \\ &= E[\int_0^s \lambda_u p_u \exp\{-\int_0^u p_v \lambda_v dv\} du \mid \mathcal{G}_t] \\ &= 1 - \exp\{-\int_0^s p_u \lambda_u du\}. \end{aligned}$$

□

## 7.2 Proof of Lemma 1

**Proof:** Note that the conditional expectation is 0 on the set  $\{\tau \leq s\}$  and that the set  $\{\tau > s\}$  is an atom in  $\mathcal{I}_s$ . For any elements  $\omega \in \mathcal{G}_t$  and  $\tilde{\omega} \in \mathcal{H}_s$ , due to the Markov property of the Poisson process,  $\tilde{\omega}$  and  $\{\tau > t\}$  are independent conditional on  $\omega$  and  $\{\tau > s\}$ . Consequently, using a version of the Bayes' rule and Proposition 1,

$$\begin{aligned} E(1_{\{\tau > t\}} \mid \mathcal{G}_t \vee \mathcal{H}_s \vee \mathcal{I}_s) &= 1_{\{\tau > s\}} E(1_{\{\tau > t\}} \mid \mathcal{G}_t \vee \mathcal{H}_s \vee \mathcal{I}_s) \\ &= 1_{\{\tau > s\}} \frac{P(\tau > t \mid \omega, \tilde{\omega})}{P(\tau > s \mid \omega, \tilde{\omega})} \\ &= 1_{\{\tau > s\}} \frac{P(\tau > t, \tilde{\omega} \mid \omega, \tau > s)}{P(\tilde{\omega} \mid \omega, \tau > s)} \\ &= 1_{\{\tau > s\}} \frac{P(\tau > t \mid \omega, \tau > s)}{P(\tau > t \mid \omega)} \\ &= 1_{\{\tau > s\}} \frac{P(\tau > s \mid \omega)}{P(\tau > t \mid \mathcal{G}_t)} \\ &= 1_{\{\tau > s\}} \frac{\exp\{-\int_0^t p_u \lambda_u du\}}{\exp\{-\int_0^s p_u \lambda_u du\}} \\ &= 1_{\{\tau > s\}} \exp\{-\int_s^t p_u \lambda_u du\}. \end{aligned}$$

□

## 7.3 Proof of Lemma 2

**Proof:** For any  $s < t$ , using the results in Lemma 1,

$$\begin{aligned} E[1_{\{t \geq \tau\}} - 1_{\{s \geq \tau\}} \mid \mathcal{F}_s] &= E[1_{\{\tau > s\}} - 1_{\{\tau > t\}} \mid \mathcal{F}_s] \\ &= 1_{\{\tau > s\}} (1 - E[\exp\{-\int_s^t p_u \lambda_u du\} \mid \mathcal{F}_s]). \end{aligned}$$



Note that conditional on  $\mathcal{G}_\infty$  and for  $x > s$  the density of default time is given by

$$\frac{\partial}{\partial x} P(\tau \leq x \mid \tau > s, \mathcal{G}_\infty) = p_x \lambda_x \exp\left\{-\int_s^x p_u \lambda_u du\right\}.$$

Hence,

$$\begin{aligned} E\left[\int_s^t p_u \lambda_u 1_{\{u < \tau\}} du \mid \mathcal{F}_s\right] &= 1_{\{\tau > s\}} E\left[\int_s^t p_u \lambda_u 1_{\{u < \tau\}} du \mid \mathcal{F}_s\right] \\ &= 1_{\{\tau > s\}} E\left[\int_s^t p_h \lambda_h \exp\left\{-\int_s^h p_u \lambda_u du\right\} \left(\int_s^h p_u \lambda_u du\right) dh \right. \\ &\quad \left. + \exp\left\{-\int_s^t p_u \lambda_u du\right\} \left(\int_s^t p_u \lambda_u du\right) \mid \mathcal{F}_s\right] \\ &= 1_{\{\tau > s\}} (1 - E[\exp\{-\int_s^t p_u \lambda_u du\} \mid \mathcal{F}_s]). \end{aligned}$$

□

## References

- [1] F. Black and M.S. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, 81(3), 637-654, 1973
- [2] D. Brigo, A. Pallavicini & R. Torresetti, *Calibration of CDO tranches with the dynamical generalized-Poisson loss model*, Working Paper, Banca IMI, 2006, available at [http://papers.ssrn.com/sol3/papers.cfm?abstract\\_id=900549](http://papers.ssrn.com/sol3/papers.cfm?abstract_id=900549).
- [3] R. Cont and A. Minca, *Recovering Portfolio Default Intensities Implied by CDO Quotes*, Mathematical Finance. doi: 10.1111/j.1467-9965.2011.00491.x, 2011
- [4] M. Davis and V. Lo, *Modeling default correlation in bond portfolios*, in C. Alexander (Ed.), *Mastering Risk Volume 2: Applications*, Prentice Hall, 141-151, 2001.
- [5] M. Davis and V. Lo, *Infectious Defaults*, Quantitative Finance, 1(4), 382-387, 2001.
- [6] D. Duffie and N. Garleanu, *Risk and valuation of collateralized debt obligations*, Financial Analysts Journal, 57(1), 41-59, 2001.
- [7] D. Duffie, L. Saita and K. Wang, *Multi-period corporate default prediction with stochastic covariates*, Journal of Financial Economics, 83(3), 635-665, 2006.
- [8] K. Giesecke and L. Goldberg, *Sequential defaults and incomplete information*, Journal of Risk, 7(1), 1-26, 2004.
- [9] K. Giesecke, L. Goldberg and X. Ding *A top down approach to multi-name credit*, Operations Research, 59(2), 283-300, 2011.
- [10] J. Gu, W.K. Ching, T.K. Siu and H. Zheng, *On Pricing Basket Credit Default Swaps*, Working Paper, 2011, available at <http://arxiv.org/abs/1204.4025>.

- [11] D. Lando, *On Cox Processes and Credit Risky Securities*, Review of Derivatives Research, 2, 99-120, 1998.
- [12] R.A. Jarrow and S.M. Turnbull, *Pricing derivatives on financial securities subject to credit risk*, Journal of Finance, 50, 53-86, 1995.
- [13] R.A. Jarrow and F. Yu, *Counterparty risk and the pricing of defaultable securities*, Journal of Finance, 56(5), 555-576, 2001.
- [14] F. Longstaff and A. Rajan, *An empirical analysis of collateralized debt obligations*, Journal of Finance, 63(2), 529-563, 2008.
- [15] D. Madan and H. Unal, *Pricing the risks of default*, Review of Derivatives Research, 2(2-3), 121-160, 1998.
- [16] R.C. Merton, *On the pricing of corporate debt: the risk structure of interest rates*, Journal of Finance, 29(2), 449-470, 1974.
- [17] I. Norros, *A compensator representation of multivariate life length distributions, with applications*, Scand. J. Stat., 13, 99-112, 1986.
- [18] L. Perko, *Differential equations and dynamical systems (3rd Edition)*, Springer-Verlag, New York, 2001
- [19] P. Schönbucher and D. Schubert, *Copula-dependent default risk in intensity models*, Working paper, University at Bonn, 2001, available at [http://www.defaultrisk.com/pp\\_corr\\_22.htm](http://www.defaultrisk.com/pp_corr_22.htm).
- [20] M. Shaked and G. Shanthikumar, *The multivariate hazard construction*, Stoch. Proc. Appl, 24, 241-258, 1987.
- [21] F. Yu, *Correlated defaults in intensity-based models*, Mathematical Finance, 17(2), 155-173, 2007.
- [22] H. Zheng and L. Jiang, *Basket CDS pricing with interacting intensities*, Finance and Stochastics, 13, 445-469, 2009.